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1995 J. Phys. A: Math. Gen. 28 L625

(http://iopscience.iop.org/0305-4470/28/23/005)

J. Phys. A: Math. Gen. 28 (1995) L625–L628. Printed in the UK

LETTER TO THE EDITOR

Existence of low-temperature critical regime in a one-dimensional Luttinger liquid with a weak link

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Received 8 August 1995

Abstract. The exact solution of the boundary sine–Gordon model is studied in the region where the scaling dimension of the boundary field $\frac{2}{3} < \Delta < 1$. The boundary contribution to the specific heat in this region scales as $C \sim T^{2\Delta^{-1}-2}$ at small temperatures.

The problem of potential scattering in Luttinger liquids has attracted a great deal of attention since Kane and Fisher [1] mapped it onto the Schmid model [2]. The latter model is described by the following action:

$$S = S_0 + M \int \mathrm{d}\tau \cos(\beta\phi(0,\tau)/2)$$
$$S_0 = \int \mathrm{d}\tau \left\{ \int_0^L \mathrm{d}x \left[\tfrac{1}{2}(\partial_\tau\phi)^2 + \tfrac{1}{2}(\partial_x\phi)^2 \right] \right\}. \tag{1}$$

Here the parameter β is determined by interactions in the bulk. It is well established that in the region $\Delta = \beta^2/2\pi < 1$ where the cosine term is relevant the model scales to the strong-coupling fixed point described by the effective action

$$S_{\mathrm{eff}} = S_0 + \tfrac{1}{2} \int \mathrm{d}\tau \left[T_B\phi(0,\tau)^2 + \tilde{T}_B \cos(2\pi\theta(0,\tau)/\beta) \right] \tag{2}$$

where

$$T_B, \tilde{T}_B \sim M(M/\Lambda)^{\Delta/(1-\Delta)}$$

and Λ is the ultraviolet cut-off.

The robustness of the effective action (2) has been proven by the instanton expansion for $\Delta \ll 1$ [1, 3], by the transformation to the equivalent-free fermion model ($\Delta = \frac{1}{2}$) [4, 5] and by the exact solution ($\Delta \leqslant \frac{1}{2}$) [6, 7]. The dual cosine is generated by the instanton expansion of the $\cos(\beta\phi/2)$-potential and is always present in the low-energy effective action. However, for $\Delta < \frac{2}{3}$ its contribution is less important than that of the ϕ^2-term. One of the purposes of this letter is to demonstrate that for the effective action $\Delta \leqslant \frac{2}{3}$ the dual cosine becomes dominant. This becomes obvious when one calculates the first correction from this the irrelevant operator to the specific heat. At large times $\tau \gg 1/T_B$, the ϕ^2-term is equivalent to the Neumann boundary condition on the ϕ-field, which, due to the duality, gives the Dirichlet boundary condition on the θ-field:

$$\phi(x=0) = 0 \qquad \partial_x\theta(x=0) = 0. \tag{3}$$

Taking this into account we get the following correlation function for the dual cosine:

$$\langle\cos(2\pi\theta(\tau)/\beta)\cos(2\pi\theta(0)/\beta)\rangle \sim \left(\frac{\pi T}{\sin\pi T\tau}\right)^{4\pi/\beta^2}. \qquad (4)$$

From here we get the following estimate for the contribution of the dual cosine to the specific heat:

$$C_{boundary} \sim T^{2/\Delta-2}(\Delta > \tfrac{2}{3}) \sim T\ln T(\Delta = \tfrac{2}{3}). \qquad (5)$$

As we see, at $\frac{2}{3} \leqslant \Delta < 1$ this contribution dominates over the linear term generated by the ϕ^2-potential [8, 9]. The rest of the letter is devoted to the Bethe ansatz solution of the model (1) in the region $\frac{2}{3} \leqslant \Delta < 1$.

Despite the fact that the boundary sine–Gordon (BSG) model (1) has been solved exactly in the sense that exact S-matrices have been found [6], the complexity of the solution in the area $\Delta > \frac{1}{2}$ has prevented it from being studied. At $\Delta = 1/\nu$, $\nu = 2, 3, \ldots$ the thermodynamic Bethe ansatz (TBA) equations were obtained by Fendley *et al* [7]:

$$\begin{aligned}\epsilon_n(v) &= \eta\delta_{n,1}\exp(-\pi v/2) + s * \ln\left[1 + e^{\epsilon_{n-1}(v)}\right]\left[1 + e^{\epsilon_{n+1}(v)}\right] \\ &\quad + \delta_{n,\nu-2}s * \ln\left[1 + e^{\epsilon_\nu(v)}\right] \qquad (n = 1, \ldots \nu - 1) \\ \epsilon_\nu(v) &= s * \ln\left[1 + e^{\epsilon_{\nu-2}(v)}\right] \end{aligned} \qquad (6)$$

$$F_{imp} = -T\int_{-\infty}^{\infty} s\left[v + \frac{2}{\pi}\ln(T_B/T)\right]\ln\left[1 + e^{\epsilon_{\nu-1}(v)}\right] \qquad (7)$$

$$\frac{1}{L}F_{bulk} = -T\int_{-\infty}^{\infty} s\left[v + \frac{2}{\pi}\ln(\Lambda/T)\right]\ln\left[1 + e^{\epsilon_1(v)}\right] \qquad (8)$$

with $\eta = +1$. Here

$$s * f(v) = \int_{-\infty}^{\infty} du \frac{f(u)}{4\cosh[\pi(v-u)/2]}.$$

Equations (6) are very similar to the equations for the conventional sine–Gordon model. In the latter case there is a duality: TBA equations are invariant under the transformation

$$\Delta \to 1 - \Delta = 1/\nu \qquad (9)$$

except for the free term in the first equation (6), which changes its sign: $\eta = -1$. Therefore we suggest that TBA equations for the BSG problem at $\Delta = 1 - 1/\nu$ are given by (6) and (7) with $\eta = -1$. We have two additional arguments in favour of this proposal. The first one is that the suggested transformation works for the anisotropic spin-$\frac{1}{2}$ Heisenberg chain with open boundary conditions [8]. It is widely believed that the latter model adequately describes the strong-coupling point of the BSG model. Indeed, at strong coupling the scattering potential becomes infinite which effectively breaks the chain. The equivalency between the Luttinger liquid of spinless fermions and the spin-$\frac{1}{2}$ Heisenberg chain is established by the Jordan–Wigner transformation. The second argument is that at $\Delta = \frac{3}{4}$ one can derive TBA equations for BSG model using its equivalency with the four-channel anisotropic Kondo model in the Toulouse limit [9]. We shall present the latter argument in detail.

The Bethe ansatz equations for an anisotropic k-channel Kondo model are given by

$$[e_k(\eta; u_a)]^N e_{2S}(\eta; u_a - 1/g) = \prod_{b=1}^{M} e_2(\eta; u_a - u_b) \qquad (10)$$

$$E = \sum_{a=1}^{M} \frac{1}{2i}\ln e_k(\eta; u_a) \qquad (11)$$

$$e_n(\eta; u) = \frac{\sinh[\eta(u - \mathrm{i}n)]}{\sinh[\eta(u + \mathrm{i}n)]} \tag{12}$$

where S is the impurity spin, g is the Kondo coupling constant, η is the anisotropy, N is the length of the system and $M = kN/2 + S - S^z$ is the number of up spins. The universal relationship between the quantities g and η and the parameters of the Hamiltonian exists only in the limit of weak anisotropy $\eta \ll 1$. The difficulty determining the Toulouse limit is resolved if we suggest that this limit corresponds to the maximal value of η at which the IR fixed point of the Kondo model still belongs to the same universality class as at $\eta \to 0$. The periodicity of the trigonometric factors in (10) suggests that the Toulouse limit is realized at $\eta = \pi/2(k + 1/\nu)$ $(\nu \to \infty)$. This limit has been considered in [10, 11] and the detailed derivation of TBA equations is given in [12]. It was shown that TBA equations in the Toulouse limit have the following form:

$$\epsilon_n = s * \ln(1 + \mathrm{e}^{\epsilon_{n-1}})(1 + \mathrm{e}^{\epsilon_{n+1}}) + \delta_{n,k-2} s * \ln(1 + \mathrm{e}^{\epsilon_k}) \qquad n = 1, \ldots, k-1 \tag{13}$$

$$\epsilon_k = s * \ln(1 + \mathrm{e}^{\epsilon_{k-2}}) - 2\exp(-\pi v/2) \tag{14}$$

$$F_{bulk} = -NT^2 \int_{-\infty}^{\infty} \mathrm{d}v\, \mathrm{e}^{-\pi v/2} \ln(1 + \mathrm{e}^{\epsilon_k}) \tag{15}$$

$$F_{boundary} = -T \int_{-\infty}^{\infty} \mathrm{d}v\, s\left(v + \frac{2}{\pi} \ln T_B/T\right) \ln(1 + \mathrm{e}^{\epsilon_{2S}}) \tag{16}$$

where the temperature T_B is defined as

$$T_B = \lim_{\nu, \Lambda \to \infty} \frac{\pi \Lambda}{2\nu} \exp(-\pi/2g)\,.$$

Executing this limit one has to be careful to keep the energy scale T_B finite. Substituting into these equations $S = \frac{1}{2}$ and $k = 4$ we reproduce the suggested equations for the BSG model with $\Delta = \frac{3}{4}$ (i.e. $\nu = 4$, $\eta = -1$). The fact that this equivalency holds only for $k = 4$ is not surprising. The conformal charge of those bulk degrees of freedom in the k-channel Kondo model which are coupled to the impurity spin is equal to [13, 14]

$$C = \frac{3k}{k+2}\,. \tag{17}$$

At $k = 4$ it is equal to 2 which corresponds to two bosonic modes interacting with the impurity. In the Toulouse limit one of these modes decouples from the impurity and the effective conformal charge becomes 1.

Now we shall calculate the IR and UV asymptotics of the boundary free energy for the BSG model with $\Delta = 1 - 1/\nu$. Analytical solutions are available for asymptotics of $\epsilon_n(v)$ at $v \to \pm\infty$ (see, for example, [15]). At large temperatures $T \gg T_B$ the free energy is determined by the asymptotics at $v \to +\infty$ where

$$(1 + \mathrm{e}^{\epsilon_{\nu-1}}) = \nu \tag{18}$$

so that we have

$$F_{boundary} \to -\frac{T}{2} \ln \nu\,. \tag{19}$$

At small temperatures the leading contribution comes from the region $v \to 0$ where $\epsilon_n(\nu \neq 1)$ are again almost constant and $\exp(\epsilon_1)$ is small. Then the corrections can be determined from the expansion in $\exp(\epsilon_1)$:

$$\begin{aligned} g_n(v) &\equiv \ln\left(1 + \mathrm{e}^{\epsilon_n(v)}\right) = g_n^{(0)} + g_n^{(1)}(v) + \cdots \\ g_n^{(0)} &= 2\ln n \; (n = 2, 3, \ldots \nu - 2) \qquad g_{\nu-1}^{(0)} = g_\nu^{(0)} \ln(\nu - 1) \\ g_n^{(1)}(v) &= \frac{1}{2n}[(n+1)a_n * g_1(v) - (n-1)a_{n+2} * g_1(v)] \end{aligned} \tag{20}$$

$$a_n(\omega) = \frac{\sinh(\nu - n)\omega}{\sinh(\nu - 1)\omega}. \tag{21}$$

Using these expressions we get the following expansion for the free energy:

$$F_{boundary} \to -\frac{T}{2}\ln(\nu - 1)$$
$$-T\int_0^\infty \mathrm{d}v\, f\left[v + \frac{2}{\pi}\ln(T_B/T)\right]\ln\left(1 + \mathrm{e}^{\epsilon_1(v)}\right) \tag{22}$$

where

$$f(\omega) = \frac{\tanh\omega}{\sinh(\nu - 1)\omega}.$$

The first term in (22) gives the finite entropy of the ground state $S(0) = -\frac{1}{2}\ln(\Delta^{-1} - 1)$. A careful analysis shows that this entropy disappears at $\Delta \leqslant \frac{1}{2}$. The ratio of the partition functions in the ultraviolet and the infrared limits is

$$Z_{UV}/Z_{IR} = \Delta^{-1/2}. \tag{23}$$

This result reproduces the expression obtained for $\Delta < \frac{1}{2}$ in [7]. At small T/T_B one can expand the second term in (22). The expansion is dictated by poles of the function $f(\omega)$. In the lowest order it gives $\delta F_{imp} \sim -T^{1+2/(\nu-1)}$ which leads to (5).

The author expresses his gratitude to P Coleman, M Evans, E Fradkin, L Ioffe and P de Sa for the valuable discussions and interest in the work.

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